RELATIVISTIC QUANTUM MECHANICS

Friday 06-04-2018, 14.00-17.00

Write your name and student number on **all** other sheets. The total number of points is 90. You can earn 5 points for each subquestion unless otherwise indicated.
Use conventions with $\hbar = c = 1$. The chiral representation of the 4×4 gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

They satisfy $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ in any representation of the γ-matrices.

PROBLEM 1: GENERAL THEORY (10pts)

1.1 Why is there a clash between the classical picture of a force (e.g. Coulomb, Newton) and the theory of special relativity? How is this remedied in field theories?

1.2 Suppose one starts with a particle of mass m and momentum p in a large box of size L. What is the appropriate description in different regimes as one decreases the box size L? Indicate the relation to different wavelengths of the particle.

PROBLEM 2: DIRAC EQUATION (30pts)

The Lagrangian for a relativistic massive fermion field ψ is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\,. \tag{1}$$

2.1 How does ψ transform under Lorentz transformations?

2.2 What is the definition of $\bar{\psi}$? Derive how it transforms under a Lorentz transformation, and compare to the transformation of ψ. (10pts)

2.3 Calculate how the combination $\bar{\psi}\gamma^\mu\psi$ transforms under an infinitesimal Lorentz transformation. Why does the Dirac Lagrangian transform as a scalar? (15pts)

PROBLEM 3: COMPLEX SCALAR FIELD (30pts)

The Lagrangian density for a relativistic massive complex scalar field ϕ is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi^* - \tfrac{1}{2} m^2 \phi \phi^* \,. \tag{2}$$

3.1 What is the momentum Π conjugate to the field ϕ?

3.2 Which commutation relations do we impose between the operators ϕ, ϕ^* and their time derivatives? Use the Schrodinger picture.

3.3 What is the expansion of the the field and its momentum in terms of creation and annihilation operators? Use the Schrodinger picture.

3.4 Under which internal symmetry is the above Lagrangian density invariant? Derive the form of the corresponding current, and prove that it is conserved. You can use the Euler-Lagrange equation in the last step. (10pts)

3.5 What does the conserved charge that follows from the previous internal symmetry count? Write out its form in terms of the ladder operators.

PROBLEM 4: NON-RELATIVISTIC LIMIT (20pts)

4.1 Write the Lagrangian density (2) in terms of $\tilde{\phi} \equiv e^{imt}\phi$. Which term(s) are negligible in the non-relativistic limit?

4.2 What is the momentum Π conjugate to the field $\tilde{\phi}$ in the non-relativistic limit?

4.3 What is the expansion of the the field and its momentum in terms of creation and annihilation operators in the non-relativistic limit? Use the Schrodinger picture.

4.4 What is the form of the position operator that one can construct in the non-relativistic limit?